

Pairs of Fan-type heavy subgraphs for pancyclicity of 2-connected graphs

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Abstract

A graph G on n vertices is Hamiltonian if it contains a spanning cycle, and pancyclic if it contains cycles of all lengths from 3 to n . In 1984, Fan presented a degree condition involving every pair of vertices at distance two for a 2-connected graph to be Hamiltonian. Motivated by Fan's result, we say that an induced subgraph H of G is f_1 -heavy if for every pair of vertices $u, v \in V(H)$, $d_H(u, v) = 2$ implies $\max\{d(u), d(v)\} \geq (n + 1)/2$. For a given graph R , G is called R - f_1 -heavy if every induced subgraph of G isomorphic to R is f_1 -heavy. In this paper we show that for a connected graph S with $S \neq P_3$ and a 2-connected claw- f_1 -heavy graph G which is not a cycle, G being S - f_1 -heavy implies G is pancyclic if $S = P_4, Z_1$ or Z_2 , where claw is $K_{1,3}$ and Z_i is the path $a_1a_2a_3 \dots a_{i+2}a_{i+3}$ plus the edge a_1a_3 . Our result partially improves a previous theorem due to Bedrossian on pancyclicity of 2-connected graphs.

Keywords: Fan-type heavy subgraph; Hamilton cycle; Cycle; Pancyclicity

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1 Introduction

We use Bondy and Murty [5] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph and H be a subgraph. Let x, y be two vertices of $V(H)$. An (x, y) -path in H is a path P connecting x and y in H . The *distance* between x and y in H , denoted by $d_H(x, y)$, is the length of a shortest (x, y) -path in H . When there is no danger of ambiguity, we use $d(x, y)$ instead of $d_G(x, y)$.

Let G be a graph on n vertices. For a given graph R , G is called R -free if G contains no induced subgraph isomorphic to R , and R - f_i -heavy if for every induced subgraph H

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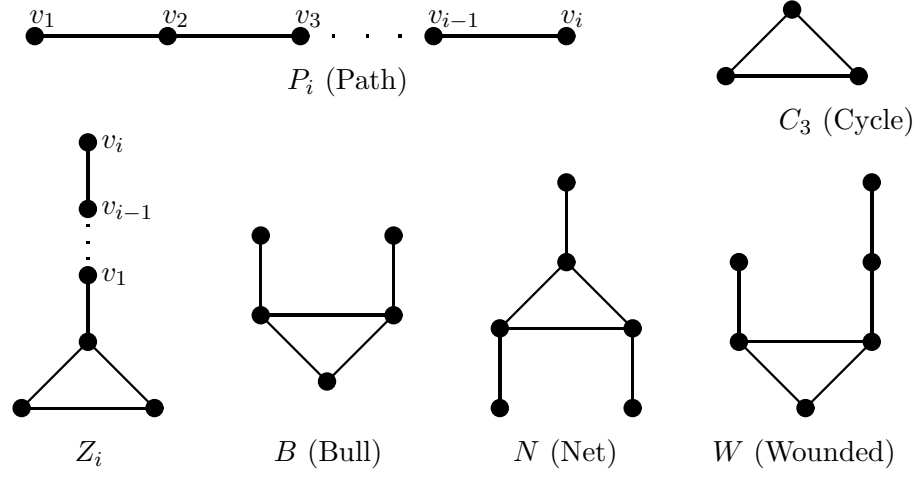


Figure 1: Graphs P_i, C_3, Z_i, B, N and W

of G isomorphic to R and every pair of vertices $u, v \in V(H)$, $d_H(u, v) = 2$ implies that $\max\{d(u), d(v)\} \geq (n + i)/2$. For a family \mathcal{R} of graphs, G is called \mathcal{R} -free (\mathcal{R} - f_i -heavy) if G is R -free (R - f_i -heavy) for each $R \in \mathcal{R}$. In particular, similar as in [9], we use R - f -heavy (\mathcal{R} - f -heavy) instead of R - f_0 -heavy (\mathcal{R} - f_0 -heavy). Note that every \mathcal{R} -free graph is also \mathcal{R} - f_1 -heavy (\mathcal{R} - f -heavy).

The bipartite graph $K_{1,3}$ is called the *claw*. We say that its (only) vertex of degree 3 is the *center* and the other vertices are its *end vertices*. In this paper, we use the terminology claw- f_1 -heavy instead of $K_{1,3}$ - f_1 -heavy.

A graph G on n vertices is said to be *Hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all vertices of G , and *pancyclic* if G contains cycles of all lengths from 3 to n . Bedrossian [1] completely characterized all the pairs of forbidden subgraphs for a 2-connected graph to be Hamiltonian and to be pancyclic.

Theorem 1 (Bedrossian [1]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$ -free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W (see Figure 1).*

Theorem 2 (Bedrossian [1]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$ -free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2 .*

In 1984, Fan [6] presented a degree condition (so-called Fan's condition) involving every pair of vertices at distance two for a 2-connected graph to be Hamiltonian.

Theorem 3 (Fan [6]). *Let G be a 2-connected graph on n vertices. If $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v such that $d(u, v) = 2$, then G is Hamiltonian.*

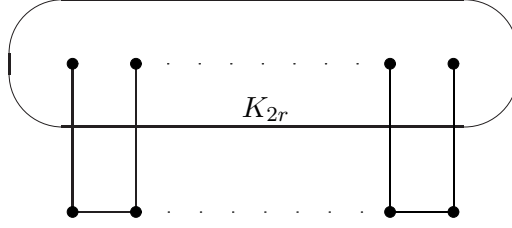


Figure 2: The Graph F_{4r}

Obviously, Fan's condition is equivalent to every 2-connected P_3 - f -heavy graph is Hamiltonian. By restricting Fan's condition to some induced subgraphs of 2-connected graphs, Ning and Zhang [9] extended Theorem 1 as follows.

Theorem 4 (Ning and Zhang [9]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$ - f -heavy implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W .*

In this paper, our aim is to find the corresponding Fan-type heavy subgraph conditions for a 2-connected graph to be pancyclic. First, from a well known result, we can deduce that every 2-connected P_3 - f_1 -heavy graph is pancyclic.

Theorem 5 (Benhocine and Wojsa [3]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is P_3 - f -heavy, then G is pancyclic unless $n = 4r$, $r > 2$, and $G = F_{4r}$ (see Figure 2), or n is even and $G = K_{n/2, n/2}$ or else $n \geq 6$ is even and $G = K_{n/2, n/2} - e$.*

Furthermore, we can see that P_3 is the only connected graph S such that every 2-connected S - f_1 -heavy graph is pancyclic. For details, see [7, Theorem 13]. So we can pose the following problem naturally.

Problem 1. Which two connected graphs R and S other than P_3 imply that every 2-connected $\{R, S\}$ - f_1 -heavy graph is pancyclic?

By Theorem 2, we know that $R = K_{1,3}$ (up to symmetry) and S must be one of Z_1, Z_2, P_4 and P_5 .

In this paper, we mainly prove the following result.

Theorem 6. *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, Z_2\}$ - f_1 -heavy, then G is pancyclic.*

As a corollary of Theorem 6, we have

Theorem 7. *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, P_4\}$ - f_1 -heavy, then G is pancyclic.*

In [2], Bedrossian et al. proved a theorem as follows.

Theorem 8 (Bedrossian, Chen and Schelp [2]). *Let G be a 2-connected graph on n vertices. If G is $\{K_{1,3}, Z_1\}$ - f -heavy, then G is pancyclic unless $G = F_{4r}$ or $G = K_{n/2, n/2}$ or $G = K_{n/2, n/2} - e$ or else G is a cycle.*

By Theorem 8, we have

Theorem 9. *Let G be a 2-connected graph which is not a cycle. If G is $\{K_{1,3}, Z_1\}$ - f_1 -heavy, then G is pancyclic.*

Combining with Theorems 6, 7 and 9, we obtain Theorem 10, which partially answers Problem 1.

Theorem 10. *Let S be a connected graph with $S \neq P_3$ and let G be a 2-connected claw- f_1 -heavy graph which is not a cycle. Then G being S - f_1 -heavy implies G is pancyclic if $S = P_4, Z_1$ or Z_2 .*

The rest of this paper is organized as follows. In Section 2, we will give additional terminology and list some useful lemmas. The proof of Theorem 6 will be postponed to Section 3.

2 Preliminaries

In this section, we first introduce some additional terminology and notation and then present four lemmas which will be used in our proof of Theorem 6.

Let G be a graph and S be a subset of $V(G)$. We use $G[S]$ to denote the subgraph of G induced by S and $G - S$ to denote $G[V(G) \setminus S]$. In particular, if $S = \{u\}$, then we use $G - u$ instead of $G - \{u\}$. If $S = \{x_i : 1 \leq i \leq 5\}$ and $G[S]$ is isomorphic to Z_2 , then we say that $\{x_1, x_2, x_3, x_4, x_5\}$ induces a Z_2 , where $x_1x_2x_3x_1$ is a triangle and x_1 is the vertex of degree 3 in $G[S]$. If $S = \{x_i : 1 \leq i \leq 4\}$ and $G[S]$ is isomorphic to $K_{1,3}$, then we say that $\{x_1, x_2, x_3, x_4\}$ induces a claw, where x_1 is the center, and x_2, x_3, x_4 are the end vertices.

Let k, l ($k < l$) be two integers. We say that G contains a k -cycle if G contains a cycle of length k , and G contains $[k, l]$ -cycles if G contains cycles of all lengths from k to l . In

particular, for a vertex $u \in V(G)$, we say that G contains a u -triangle if G contains the cycle $uxyu$, where $x, y \in V(G)$.

A vertex v of a graph G on n vertices is called *heavy* if $d(v) \geq n/2$, and *super-heavy* if $d(v) \geq (n+1)/2$. For two vertices u, v of G , $\{u, v\}$ is called a *heavy-pair* if $d(u) + d(v) \geq n$ and a *super-heavy pair* if $d(u) + d(v) \geq n + 1$.

Lemma 1 (Benhocine and Wojda [3]). *Let G be a graph on $n \geq 4$ vertices and C be a cycle of length $n - 1$ in G . If $d(x) \geq n/2$ for the vertex $x \in V(G) \setminus V(C)$, then G is pancyclic.*

Lemma 2 (Bondy [4]). *Let G be a graph on n vertices with a Hamilton cycle C . If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d(x) + d(y) \geq n + 1$, then G is pancyclic.*

Lemma 3 (Hakimi and Schmeichel [10]). *Let G be a graph on n vertices with a Hamilton cycle C . If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d(x) + d(y) \geq n$, then G is pancyclic unless G is bipartite or else G is missing only an $(n - 1)$ -cycle.*

Lemma 4 (Ferrara, Jacobson and Harris [8]). *Let G be a graph on n vertices with a Hamilton cycle C . If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 2$ and $d(x) + d(y) \geq n + 1$, then G is pancyclic.*

3 Proof of Theorem 6

We prove Theorem 6 by contradiction. Suppose that G satisfies the condition of Theorem 6 but is not pancyclic. Since the result is easy to verify for $3 \leq n \leq 5$, we assume that $n \geq 6$.

If G is $\{K_{1,3}, Z_2\}$ -free, then by Theorem 2, G is pancyclic. Thus we assume that G contains an induced claw or an induced Z_2 . Therefore, there is a super-heavy vertex, say $u \in V(G)$. Set $G' = G - u$. Since G is $\{K_{1,3}, Z_2\}$ - f_1 -heavy, G' is $\{K_{1,3}, Z_2\}$ - f -heavy. If G' is 2-connected, then by Theorem 4, G' is Hamiltonian. Hence G is pancyclic by Lemma 1. Now, it will be assumed that G' is not 2-connected. Then there exists a vertex $v \in V(G)$ ($v \neq u$) such that $G - \{u, v\}$ is not connected. By Theorem 4, G is Hamiltonian. Hence $G - \{u, v\}$ consists of two components H_1 and H_2 . Without loss of generality, we assume that $|V(H_1)| \leq |V(H_2)|$, where $V(H_1) = \{x_1, x_2, \dots, x_{h_1}\}$ and $V(H_2) = \{y_1, y_2, \dots, y_{h_2}\}$. Let $C = uy_1 \cdots y_{h_2} vx_{h_1} \cdots x_1 u$ be a Hamilton cycle with the given orientation. In the following, for any two vertices $w_1, w_2 \in V(C)$, we use $C[w_1, w_2]$ to denote the segment of C from w_1 to w_2 along the orientation. Set $G_1 = G[V(H_1) \cup \{u\}]$ and $G_2 = G[V(H_2) \cup \{u\}]$.

Claim 1. There are no super-heavy vertices in H_1 .

Proof. For any vertex $x \in V(H_1)$, x is adjacent to at most u, v and all the vertices in H_1 except for itself. Therefore, $d(x) \leq d_{H_1}(x) + 2 \leq h_1 - 1 + 2 \leq n/2$. Hence H_1 contains no super-heavy vertices. \square

Claim 2. $N_{G_2}(u) \setminus \{y_1\} \subseteq N(y_1)$.

Proof. If there exists a vertex $y_i \in N_{G_2}(u) \setminus \{y_1\}$ such that $y_i y_1 \notin E(G)$, then $\{u; x_1, y_1, y_i\}$ induces a claw. By Claim 1, x_1 is not super-heavy. Since G is claw- f_1 -heavy, y_1 is super-heavy. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$. By Lemma 2, G is pancyclic. \square

Claim 3. There are no super-heavy pairs with distance one or two along the orientation of a Hamilton cycle in G .

Proof. Suppose not. By Lemma 2 or 4, G is pancyclic. \square

Case 1. $h_1 = 1$.

Subcase 1.1. $uv \in E(G)$.

Note that G cannot be bipartite or missing an $(n - 1)$ cycle, so if Lemma 3 applies to G then G is pancyclic. If u is adjacent to every vertex in C , then G is pancyclic. Now we can choose a vertex $y_i \in N_{G_2}(u)$ such that $uy_{i+1} \notin E(G)$. Let y_j be the first vertex on $C[y_i, y_{h_2}]$ such that $uy_{j+1} \in E(G)$, where assume that $y_{h_2+1} = v$. Obviously, $j \geq i + 1$.

Claim 4. $i \geq 2$.

Proof. Assume there exists $y \in V(H_2)$ such that $y_1 y \in E(G)$ and $uy \notin E(G)$. By Claim 2, we have $N_{G_2}(u) \setminus \{y_1\} \subset N(y_1)$. Since $d(u) \geq (n + 1)/2$ and $u, y \in N(y_1) \setminus N(u)$, $d(y_1) \geq d(u) - 3 + 2 \geq (n - 1)/2$. Therefore, $\{u, y_1\}$ is a heavy-pair such that $d_C(u, y_1) = 1$. By Lemma 3, G is pancyclic. Also, since $y_1 y_2 \in E(G)$, then $uy_2 \in E(G)$ and $i \geq 2$. \square

Next we assume that $i \leq h_2 - 2$. Note that $y_i, y_{i+1}, y_{i+2} \in C[y_2, y_{h_2}]$.

Claim 5. $j \geq i + 2$.

Proof. Assume that $j = i + 1$. First, we have $uy_i, uy_{i+2} \in E(G)$ and $uy_{i+1} \notin E(G)$.

If $y_i y_{i+2} \notin E(G)$, then $\{u; x_1, y_i, y_{i+2}\}$ induces a claw. Since $d(x_1) = 2 < (n + 1)/2$ and G is claw- f_1 -heavy, $\{y_i, y_{i+2}\}$ is a super-heavy pair such that $d_C(y_i, y_{i+2}) = 2$, which contradicts to Claim 3.

Now assume that $y_i y_{i+2} \in E(G)$. If $y_1 y_{i+1} \in E(G)$, then it follows $d(y_1) \geq (n-1)/2$ from Claim 2. Hence $\{u, y_1\}$ is a heavy pair with $d_C(u, y_1) = 1$, and G is pancyclic by Lemma 3. Therefore, $y_1 y_{i+1} \notin E(G)$. We set $G' = G - y_i$. Clearly, $C' = C[y_{i+2}, y_i] y_i y_{i+2}$ is a Hamilton cycle in G' . Moreover, u, y_1 satisfy that $d_{G'}(u) + d_{G'}(y_1) = d(u) + d(y_1) \geq (n+1)/2 + (n-3)/2 = n-1$ and $d_{G'}(u, y_1) = 1$. By Lemma 3, G' is pancyclic. Together with the cycle C , G is pancyclic. \square

By Claim 5, we obtain $u y_{i+2} \notin E(G)$.

Claim 6. $vy_{i+1} \in E(G)$.

Proof. Assume that $vy_{i+1} \notin E(G)$.

Claim 6.1. $vy_{i+2} \notin E(G)$.

Proof. Assume that $vy_{i+2} \in E(G)$. Then $\{v, x_1, u; y_{i+2}, y_{i+1}\}$ induces a Z_2 . If v is a super-heavy vertex, then $\{u, v\}$ is a super-heavy pair such that $d_C(u, v) = 2$, contradicting to Claim 3. Now assume that v is not super-heavy. Note that x_1 is not super-heavy. Since G is Z_2 - f_1 -heavy, $\{y_{i+1}, y_{i+2}\}$ is a super-heavy pair such that $d_C(y_i, y_{i+1}) = 1$, contradicting to Claim 3. \square

Claim 6.2. $vy_i \notin E(G)$.

Proof. Assume that $vy_i \in E(G)$. By Claim 6.1, we have $vy_{i+2} \notin E(G)$. Note that $vy_{i+1} \notin E(G)$ by the initial hypothesis. If $y_i y_{i+2} \notin E(G)$, then $\{y_i, u, v; y_{i+1}, y_{i+2}\}$ induces a Z_2 . Since v is not super-heavy, y_{i+1} is super-heavy. Hence either $\{y_{i+1}, y_{i+2}\}$ or $\{y_{i+1}, y_i\}$ is a super-heavy pair, a contradiction by Claim 3. If $y_i y_{i+2} \in E(G)$, then $\{y_i, y_{i+1}, y_{i+2}; v, x_1\}$ induces a Z_2 . Since v is not super-heavy, $\{y_{i+1}, y_{i+2}\}$ is a super-heavy pair such that $d_C(y_i, y_{i+1}) = 1$, a contradiction by Claim 3. \square

Claim 6.3. y_i is super-heavy.

Proof. By Claims 6.2 and the initial hypothesis, we have $vy_i \notin E(G)$ and $vy_{i+1} \notin E(G)$. Since $\{u, v, x_1; y_i, y_{i+1}\}$ induces a Z_2 and x_1 is not super-heavy, y_i is super-heavy. \square

By Claim 4, we have $i \geq 2$, and this implies y_{i-1} is well-defined.

Claim 6.4. $y_{i-1} y_{i+1} \notin E(G)$, $u y_{i-1} \in E(G)$, $y_i y_{i+2} \notin E(G)$ and $y_{i-1} y_{i+2} \notin E(G)$.

Proof. By Claim 6.3, y_i is super-heavy. If $y_{i-1} y_{i+1} \in E(G)$, then G is pancyclic by Lemma 1.

If $uy_{i-1} \notin E(G)$, then $\{y_i; y_{i-1}, y_{i+1}, u\}$ induces a claw. Hence either y_{i-1} or y_{i+1} is super-heavy. Therefore, either $\{y_{i-1}, y_i\}$ or $\{y_i, y_{i+1}\}$ is a super-heavy pair such that $d_C(y_{i-1}, y_i) = d_C(y_i, y_{i+1}) = 1$, a contradiction by Claim 3.

By Claim 2 and Lemma 3, $y_1y_{i+1} \notin E(G)$. If $y_iy_{i+2} \in E(G)$, then set $G' = G - y_{i+1}$. Now $C' = vx_1uy_1 \dots y_iy_{i+2} \dots y_{h_2}v$ is a Hamilton cycle in G' , and $d_{G'}(u) + d_{G'}(y_1) \geq n - 1 = |G'|$ by Claim 2. By Lemma 3, G' is either pancyclic, bipartite, or missing only an $(n - 2)$ -cycle. Since $C' = vx_1uy_1 \dots y_iy_{i+2} \dots y_{h_2}v$ is an $(n - 1)$ -cycle and $C'' = vuy_1 \dots y_iy_{i+2} \dots y_{h_2}v$ is an $(n - 2)$ -cycle in G' , G' is pancyclic. Therefore, G is pancyclic.

If $y_{i-1}y_{i+2} \in E(G)$, then set $G' = G - y_{i+1}$. Now $C' = uy_1 \dots y_{i-1}y_{i+2} \dots y_{h_2}vx_1u$ is a Hamilton cycle in $G'' = G' - y_i$ and $d_{G'}(y_i) \geq (n - 1)/2 = |G'|/2$. By Lemma 1, G' is pancyclic. Together with the cycle C , G is pancyclic. \square

By Claim 6.4, $\{y_i, u, y_{i-1}; y_{i+1}, y_{i+2}\}$ induces a Z_2 . Since G is Z_2 - f_1 -heavy, either y_{i-1} or y_{i+1} is super-heavy. Then either $\{y_{i-1}, y_i\}$ or $\{y_{i+1}, y_i\}$ is a super-heavy pair such that $d_C(y_{i-1}, y_i) = d_C(y_{i+1}, y_i) = 1$. By Claim 3, a contradiction. \square

Claim 7. For any $k \in \{i + 1, \dots, j\}$, $vy_k \in E(G)$.

Proof. By Claim 6, we have $vy_{i+1} \in E(G)$. Now we show that $vy_k \in E(G)$ for any $k \in \{i + 2, \dots, j\}$. Otherwise, assume that y_t is the first vertex on $C[y_{i+2}, y_j]$ such that $vy_t \notin E(G)$. Note that for any $k \in \{i + 1, \dots, j\}$, $uy_k \notin E(G)$. We have $uy_{t-1}, uy_t \notin E(G)$. Then $\{v, x_1, u; y_{t-1}, y_t\}$ induces a Z_2 . Since x_1, v are not super-heavy, $\{y_{t-1}, y_t\}$ is a super-heavy pair such that $d_C(y_{t-1}, y_t) = 1$. By Claim 3, a contradiction, hence $vy_k \in E(G)$. \square

Note that since $j \geq i + 2$ and i could be selected to be $\leq h_2 - 2$, then if $(j + 1) \leq h_2 - 2$, let $i = j + 1$ and repeat the previous arguments to conclude that for any vertex $y \in \{y_2, y_3, \dots, y_{h_2-2}\}$ such that $uy \notin E(G)$, we have $vy \in E(G)$. Hence $d_{C[y_1, y_{h_2-2}]}(u) + d_{C[y_1, y_{h_2-2}]}(v) \geq h_2 - 2$. If $uy_{h_2-1} \in E(G)$ or $vy_{h_2-1} \in E(G)$ or $uy_{h_2} \in E(G)$, then $d_{G_2}(u) + d_{G_2}(v) \geq h_2 = n - 3$. This implies that $d(u) + d(v) \geq n + 1$. By Claim 3, a contradiction. Otherwise, assume that $uy_{h_2-1}, uy_{h_2} \notin E(G)$ and $vy_{h_2-1} \notin E(G)$. Then $\{v, x_1, u; y_{h_2}, y_{h_2-1}\}$ induces a Z_2 . It follows that $\{y_{h_2}, y_{h_2-1}\}$ is a super-heavy pair such that $d_C(y_{h_2-1}, y_{h_2}) = 1$, contradicting to Claim 3.

Subcase 1.2. $uv \notin E(G)$.

By Claim 2, $N_{G_2}(u) \setminus \{y_1\} \subseteq N(y_1)$. If $uy_2 \notin E(G)$, then since u is super-heavy and $u, y_2 \in N(y_1) \setminus N(u)$, y_1 is super-heavy. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$, a contradiction by Claim 3. If $uy_2 \in E(G)$, then we have $d(y_1) \geq (n - 1)/2$ and $\{u, y_1\}$ is a heavy-pair such that $d_C(u, y_1) = 1$. By Lemma 3, G is either pancyclic,

bipartite, or missing only an $(n-1)$ -cycle. The cycle uy_1y_2u (a triangle) is odd, so G is not bipartite. Since $C' = ux_1vy_{h_2}, \dots, y_2u$ is an $(n-1)$ -cycle, G is pancyclic.

Case 2. $h_1 \geq 2$.

Subcase 2.1. G_1 contains a u -triangle.

Without loss of generality, we denote a u -triangle in G_1 by $ux_kx_{k'}u$ where $k < k'$.

Subsubcase 2.1.1. u is not adjacent to every vertex of H_2 .

Let $y_i \in V(H_2)$ be the vertex such that $uy_i \notin E(G)$ and i is as small as possible. Note that $\{u, x_k, x_{k'}; y_{i-1}, y_i\}$ induces a Z_2 . By Claim 1, y_{i-1} is super heavy. So if $i = 2$ then $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$, a contradiction by Claim 3. Therefore $i \geq 3$ and $uy_2 \in E(G)$.

If there exists $t \in \{1, 2, \dots, h_1 - 1\}$ such that $ux_t \in E(G)$ and $ux_{t+1} \notin E(G)$, then $\{u, y_1, y_2; x_t, x_{t+1}\}$ induces a Z_2 . Note that x_t is not super-heavy. Since G is Z_2 - f_1 -heavy, y_1 is super-heavy. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$, contradicting to Claim 3. Therefore, u is adjacent to every vertex of H_1 . Note that $C' = ux_1 \cdots x_i u$ is an $(i+1)$ -cycle, where $2 \leq i \leq h_1$, and G contains $[3, h_1 + 1]$ -cycles. If $i = h_2$, then u is adjacent to every vertex of H_2 other than y_{h_2} . It follows G contains $[h_1 + 4, n]$ -cycles. Furthermore, $C' = ux_2 \cdots x_{h_1}vy_{h_2}y_{h_2-1}u$ is an $(h_1 + 3)$ -cycle. If $h_1 \geq 3$, then $C' = ux_3 \cdots x_{h_1}vy_{h_2}y_{h_2-1}u$ is an $(h_1 + 2)$ -cycle, and G is pancyclic. If $h_1 = 2$ and $h_2 \geq 4$, then we can easily find a 4-cycle in G , and G is pancyclic. If $h_1 = 2$ and $h_2 = 2$ or 3, then $n = 6$ or 7. In these two cases, the result is easy to verify.

Now we suppose that $3 \leq i \leq h_2 - 1$ and try to get a contradiction. If there exists $y_k \in N_{G_2}(u)$ such that $y_k y_{i-2} \notin E(G)$ and $y_k \neq y_{i-2}$, then $\{u; x_1, y_k, y_{i-2}\}$ induces a claw. Since G is claw- f_1 -heavy and x_1 is not super-heavy, y_{i-2} is super-heavy. Therefore, $\{y_{i-2}, y_{i-1}\}$ is a super-heavy pair such that $d_C(y_{i-2}, y_{i-1}) = 1$, a contradiction by Claim 3. So, $N_{G_2}(u) \setminus \{y_{i-2}\} \subseteq N(y_{i-2})$.

If $uv \in E(G)$, then we set $G' = G - V(H_1)$. Since $N(u) \cup \{u\} \setminus (V(H_1) \cup \{v, y_{i-2}\}) \subseteq N(y_{i-2})$, we have $d(y_{i-2}) \geq d(u) + 1 - h_1 - 2 \geq (n+1)/2 - h_1 - 1$. Furthermore, we obtain $d_{G'}(y_{i-2}) + d_{G'}(y_{i-1}) = d(y_{i-2}) + d(y_{i-1}) \geq n - h_1 = |G'|$. Let $C' = uv y_{h_2} \cdots y_1 u$. Then C' is a Hamilton cycle in G' and $d_{C'}(y_{i-2}, y_{i-1}) = 1$. By Lemma 3, G' is either pancyclic, bipartite, or missing only a $(|G'| - 1)$ -cycle. But G' contains the triangle uy_1y_2u , hence it is not bipartite. Note that G contains the cycle $C'' = uv y_{h_2} \cdots y_2 u$ of length $|G'| - 1$. Hence G' is pancyclic, and this implies that G contains $[3, |G'|]$ -cycles. Since u is adjacent to every vertex of H_1 , G contains $[|G'| + 1, n]$ -cycles. Hence G is pancyclic.

If $uv \notin E(G)$, then we set $G' = G - (V(H_1) \setminus \{x_{h_1}\})$. Now we have $d(y_{i-2}) \geq d(u) - h_1 - 1 + 1 \geq (n+1)/2 - h_1$. And we obtain $d_{G'}(y_{i-2}) + d_{G'}(y_{i-1}) \geq d(y_{i-2}) + d(y_{i-1}) \geq n+1 - h_1 = |G'|$. Similarly, we can prove that G is pancyclic.

Subsubcase 2.1.2. u is adjacent to every vertex of H_2 .

Note that uy_1y_2u is a u -triangle. If there exists a vertex $x_t \in V(H_1)$ such that $ux_t \in E(G)$ and $ux_{t+1} \notin E(G)$, then $\{u, y_1, y_2; x_t, x_{t+1}\}$ induces a Z_2 . This implies that y_1 is super-heavy. Hence $\{u, y_1\}$ is a super-heavy pair such that $d_C(u, y_1) = 1$, a contradiction by Claim 3. If u is adjacent to every vertex in H_1 , then u is adjacent to all vertices of $V(G) \setminus \{u, v\}$. This implies that $d(u) \geq n-2$, and $d(u) + d(y_1) \geq n$. By Lemma 3, G is either pancyclic, bipartite, or missing only an $(n-1)$ -cycle. Since u is adjacent to every vertex in H_2 , G is neither bipartite nor missing $(n-1)$ -cycles. It follows that G is pancyclic.

Subcase 2.2. G_1 contains no u -triangles.

We first show that $N_{G_1}(u) = \{x_1\}$. Suppose not. If there is a vertex $x \in N_{G_1}(u)$ such that $x \neq x_1$, then since G_1 contains no u -triangles, we have $xx_1 \notin E(G)$. Now $\{u; x, x_1, y_1\}$ induces a claw. It follows that either x or x_1 is super-heavy, which contradicts to Claim 1.

If there exist two consecutive vertices, say $y_i, y_{i+1} \in V(H_2)$, such that $uy_i, uy_{i+1} \in E(G)$, then $\{u, y_i, y_{i+1}; x_1, x_2\}$ induce a Z_2 . Hence $\{y_i, y_{i+1}\}$ is a super-heavy pair such that $d_C(y_i, y_{i+1}) = 1$, a contradiction by Claim 3.

Therefore for any $y_i \in V(H_2) \setminus \{y_{h_2}\}$, $|\{uy_i, uy_{i+1}\} \cap E(G)| \leq 1$. This implies that u is adjacent to only one vertex x_1 in H_1 and at most $(h_1+1)/2$ vertices in H_2 and maybe adjacent to v or not. Hence we have $(n+1)/2 \leq d(u) \leq 1 + 1 + (h_2+1)/2$. This implies that $h_2 \geq n-4$. Noting that $h_2 = n-2-h_1 \leq n-2-2 = n-4$, we have $h_2 = n-4, h_1 = 2$, $uv \in E(G)$ and $N_{G_2}(u) = \{y_{2k+1} : k = 0, 1, \dots, (n-5)/2\}$, where n is odd.

If $y_1y_3 \notin E(G)$, then $\{u; x_1, y_1, y_3\}$ induces a claw. Since G is claw- f_1 -heavy, $\{y_1, y_3\}$ is a super-heavy pair such that $d_C(y_1, y_3) = 2$. By Claim 3, a contradiction.

If $y_1y_3 \in E(G)$, then $\{u, y_1, y_3; x_1, x_2\}$ induces a Z_2 . Since G is Z_2 - f_1 -heavy, $\{y_1, y_3\}$ is a super-heavy pair such that $d_C(y_1, y_3) = 2$. By Claim 3, also a contradiction.

The proof is complete. \square

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